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On Perfectness in Topological Algebras

Reporte de investigación

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Abstract

Some topological algebras can be represented as the projective limit of a projective system of algebras of some type. This is the case of complete m-convex algebras and their Arens-Michael decomposition. In this paper we study the notion of **perfectness** applied to a topological algebra and prove that all complete m-pseudoconvex algebras are perfect. We also prove that this property is preserved under finite products.

1 Introduction

The main purpose of this paper is to study the notion of *perfectness* applied to topological algebras. This is a notion concerning projective limits.

The term *perfect* has been applied to several notions in the frame of Topological Algebras but referring to very different concepts. Apostol [2] used it to describe the density of a certain ideal in locally m-convex *-algebras, whereas Shultz [13] and Archbold [3] used it to describe certain subalgebra of a C^*-algebra related to its states. Others authors have also used the term in other contexts.

Here, by **perfectness** we will mean a property inherent to a projective system of topological algebras and its corresponding projective limit topological algebra.

The most important example of a representation of an algebra in terms of other type of algebras is the well known *Arens-Michael decomposition* which states that a complete m-convex algebra is isomorphic to the projective limit algebra of a projective system of Banach algebras (see [11, Chap. 3, Sect. 3]). Several generalizations and particular cases of this tool have been provided in the literature by many authors. We can mention the *generalized Arens-Michael decomposition* for complete m-pseudoconvex algebras given by Balachandran [4, Th. 4.5.3, p. 202] and Abel [1], the definition and study of locally C^*-algebras by Inoue [10] and the definition and study of locally H^*-algebras by Haralampidou [7] and the authors [8].

In all these cases, the projective system is constructed via the quotient algebras of the original algebra modulo the kernel of a seminorm (pseudo-seminorm or C^*-seminorm). Nevertheless, it is interesting to study the situation in which the projective system does not necessarily arise from a certain family of seminorms. Such is the case in Phillips [12] where the author considers algebras which are projective limit of a projective system of C^*-algebras. He defines *pro-C^*-algebras* slightly different than Inoue does. Nevertheless, he points out that they are "essentially" equivalent to *locally C^*-algebras* in the sense of Inoue.

In the before mentioned paper, in order to describe the multiplier algebra of a given pro-C^*-algebra, Philips has to assume that certain homomorphisms are onto [12, Th. 3.14]. This approach was also the one the authors took in a previous paper related to the description of the multiplier algebra of a complete m-pseudoconvex algebra [9]. The assumption of Philips relates to the main concept we deal with in this paper, the *perfectness* of the projective system. The concept of perfectness considered in this paper was introduced by Haralampidou in [7]. Due to the diversity of notions refered as "perfectness", we will use

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the term *projective system perfectness* and denote it just by **ps-perfectness.** We think that this term reflects the essence of the property pointed out by Phillips.

By a *topological algebra* we mean a topological vector space (A, τ) in which it is defined an associative multiplication that turns A into an algebra and that is *separately continuous* with respect to τ.

Throughout this paper, (A, τ) will denote just a topological algebra over the field $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

2 Projective systems in Topological Algebras

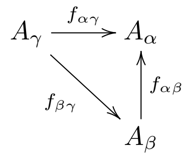
Recall that a partially ordered set $(\Lambda, \leq)$ is **directed** if for each pair of elements α and β in Λ there is an element γ in Λ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition 1. *A **projective system** of topological algebras is a pair*

$$(\{A_\alpha\}_{\alpha \in \Lambda}, \{f_{\alpha\beta}\}_{\alpha \leq \beta})$$

where Λ is a directed set, for each $\alpha \in \Lambda$, $A_\alpha = (A_\alpha, \tau_\alpha)$ is a topological algebra, $f_{\alpha\beta} : A_\beta \to A_\alpha$ is a homomorphism of topological algebras whenever $\alpha \leq \beta$ in Λ, and they satisfy the following conditions:

- *$f_{\alpha\alpha} = Id_{A_\alpha}$, the identity map in A_α, and*
- *$\alpha \leq \beta \leq \gamma$ in Λ implies $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ (see diagram)*

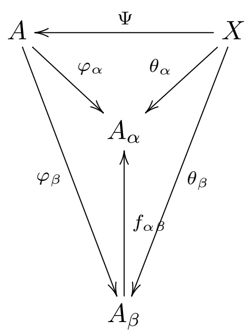


To each projective system $(\{A_\alpha\}_{\alpha \in \Lambda}, \{f_{\alpha\beta}\}_{\alpha \leq \beta})$, there corresponds its **projective limit topological algebra,** denoted by $\varprojlim A_\alpha$, which can be defined via a universal property in the following way: $\varprojlim A_\alpha$ is a pair $(A, \{\varphi_\alpha\}_{\alpha \in \Lambda})$ where $A = (A, \tau)$ is a topological algebra and, for each $\alpha \in \Lambda$, $\varphi_\alpha : A \to A_\alpha$ is a homomorphism of topological algebras that are subjected to the following conditions:

- If $\alpha \leq \beta$ in Λ, then $f_{\alpha\beta} \circ \varphi_\beta = \varphi_\alpha$.
- If $(X, \{\theta_\alpha\}_{\alpha \in \Lambda})$ is another pair satisfying the same properties, that is, $X = (X, \sigma)$ is a topological algebra and $\theta_\alpha : X \to A_\alpha$ is a homomorphism of topological algebras such that $\alpha \leq \beta$ implies $f_{\alpha\beta} \circ \theta_\beta = \theta_\alpha$, then there exists a unique topological algebra homomorphism

  $$\Psi : X \to A$$

  such that $\varphi_\alpha \circ \Psi = \theta_\alpha$ for each $\alpha \in \Lambda$ (see diagram).



Let us remark that in the Category of Topological Algebras, projective limits always exist. By the universal property, projective limits are unique up to topological algebras isomorphism. The projective limit of a projective system can be realized in the following way:

Let $(\{A_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha\beta}\}_{\alpha\leq\beta})$ be a given projective system of topological algebras. Consider the cartesian product topological algebra $\prod\limits_{\alpha\in\Lambda} A_\alpha$ as well as the topological subalgebra

$$A = \left\{ (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} A_\alpha : f_{\alpha\beta}(x_\beta) = x_\alpha \text{ if } \alpha \leq \beta \right\}.$$

Denote by π_α the canonical projection homomorphism (of topological algebras)

$$\pi_\alpha : \prod_{\alpha\in\Lambda} A_\alpha \to A_\alpha$$

and define, for each $\alpha \in \Lambda$,

$$\varphi_\alpha : A \to A_a \text{ as } \varphi_\alpha = \pi_\alpha \mid_A$$

the restriction homomorphism.

It is known that $(A, \{\varphi_\alpha\}_{\alpha\in\Lambda})$ is the projective limit of $(\{A_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha\beta}\}_{\alpha\leq\beta})$ in the sense of the above definition (see [11, Page 83]).

Definition 2. *A projective system $(\{A_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha\beta}\}_{\alpha\leq\beta})$ of topological algebras is called* ***perfect*** *if the morphisms*

$$\varphi_\alpha : \varprojlim \mathcal{A}_\alpha \to A_\alpha$$

are onto (for every $\alpha \in \Lambda$).

The respective projective limit topological algebra $\varprojlim A_\alpha$ is then called a ***ps-perfect (projective limit topological) algebra****.*

A pair $((\{A_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha\beta}\}_{\alpha\leq\beta}), A)$, where $(\{A_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha\beta}\}_{\alpha\leq\beta})$ is a projective system of topological algebras and A is a topological algebra is named ***ps-perfect****, if the system is perfect and A is algebraically and topologically isomorphic to the respective ps-perfect projective limit algebra.*

A terminological comment. According to the previous definition, the term *ps-perfect algebra* comes after the fact that such a topological algebra has to do with some perfect projective system of topological algebras. So, that notion depends on the projective system. This means that a topological algebra can be ps-perfect with respect to a projective system, but not a ps-perfect one with respect to another projective system.

Example 3. *This is an example of a non-perfect projective system. It can be found in [12, Example 2.14, p. 174].*

Let Y be a regular space which is not a completely regular space. For an example of such a space, see [5, Example 3, Section VII.7]. So in Y there are points that cannot be separated by a continuous function.

Consider $C(Y)$, the algebra of all continuous complex-valued functions defined on Y. It is known that $C(Y) = \varprojlim C(K)$, where K runs through the set of all compact subsets of Y ordered by inclusion and that all the restriction maps $f_{LK} : C(K) \to C(L)$ are surjective (K and L compact subsets of Y such that $L \subseteq K$).

Nevertheless, not all the maps $C(Y) \to C(K)$ are surjective. For example, take a and b two points in Y that cannot be separated by a continuous function and let $K = \{a, b\}$. Then the map $C(Y) \to C(K)$ is not surjective.

Remark 4. *Since the notion of ps-perfectness is defined via a universal property, it is clear that this notion is isomorphism-invariant. That is, if A is a ps-perfect topological algebra with respect to some projective system and $B \cong A$ (as topological algebras), then B is also a ps-perfect topological algebra with respect to the same projective system.*

3 The Arens-Michael decomposition

Let us recall the generalized Arens-Michael decomposition for an m -pseudo-convex algebra.

Let $(A, \{p_\alpha\}_{\alpha\in\Lambda}\}$ be an m-pseudo-convex algebra whose topology is defined by the family of pseudo-seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$. Then, for each $\alpha \in \Lambda$, let

$$\begin{array}{rcl}\rho_\alpha & : & A \to A/\ker p_\alpha = A_\alpha \\ x & \longmapsto & x + \ker p_\alpha \doteq x_\alpha\end{array}$$

denote the continuous canonical projection from the algebra A to the pseudo-normed algebra $(A_\alpha, \overset{\bullet}{p}_\alpha)$ where $\overset{\bullet}{p}_\alpha(x_\alpha) = p_\alpha(x)$. Moreover, let us denote by $(\widetilde{A_\alpha}, \|\cdot\|_\alpha)$ the completion of A_α with respect to that pseudo-norm.

Endow the set Λ with a partial ordering stating that $\alpha \le \beta$ if $p_\alpha(x) \le p_\beta(x)$ for every $x \in A$. Then we have that $\ker p_\beta \subseteq \ker p_\alpha$ if $\alpha \le \beta$. If the original set of pseudo-seminorms $\{p_\alpha\}_{\alpha\in\Lambda}$ is saturated (which can always be assumed, and in fact we will assume it to be so), then the partially ordered set $(\Lambda, \le)$ is directed.

When $\alpha \le \beta$, consider the well-defined continuous surjective homomorphism

$$\begin{array}{rcl}f_{\alpha\beta} & : & A_\beta \to A_\alpha \\ x_\beta & \longmapsto & f_{\alpha\beta}(x_\beta) \doteq x_\alpha\end{array}$$

and consider also its continuous extension

$$\widetilde{f_{\alpha\beta}} : \widetilde{A_\beta} \to \widetilde{A_\alpha}.$$

It is easy to verify that we have two projective systems of topological algebras $(\{A_\alpha\}_{\alpha\in\Lambda}, \{f_{\alpha\beta}\}_{\alpha\le\beta})$ and $(\{\widetilde{A_\alpha}\}_{\alpha\in\Lambda}, \{\widetilde{f_{\alpha\beta}}\}_{\alpha\le\beta})$, the first consisting of pseudo-normed algebras and the second consisting of k-Banach algebras (complete pseudo-normed algebras).

The generalized Arens-Michael decomposition states that, if A is complete, then

$$A \cong \varprojlim A_\alpha \cong \varprojlim \widetilde{A_\alpha}$$

up to topological algebras isomorphisms.

If the algebra is not complete the situation is

$$A \hookrightarrow \varprojlim A_\alpha \hookrightarrow \varprojlim \widetilde{A_\alpha} \cong \widetilde{A}$$

up to topological algebras monomorphisms or isomorphism (see [11, Page 88]).

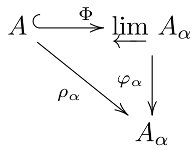
Proposition 5. *The projective system arising from the generalized Arens-Michael analysis for any locally m-pseudoconvex algebra is perfect.*

Proof. Let $(A, \{p_\alpha\}_{\alpha\in\Lambda})$ be an m-pseudo-convex algebra (not necessarily complete). Let

$$\Phi : A \to \varprojlim A_\alpha$$

be the monomorphism in the generalized Arens-Michael analysis for A. Let $\rho_\alpha : A \to A/\ker p_\alpha$ be the canonical projection defined by $\rho_\alpha(x) = x + \ker p_\alpha \doteq x_\alpha$. Let also $\varphi_\alpha : A \to A_a$ be the restriction homomorphism of the canonical projection $\pi_\alpha : \prod_{\alpha\in\Lambda} A_\alpha \to A_\alpha$ to the limit algebra $\varprojlim A_\alpha = \{(x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} A_\alpha : f_{\alpha\beta}(x_\beta) = x_\alpha \text{ if } \alpha \le \beta \text{ in } \Lambda\}$.

Note that, for $x \in A$, $\Phi(x) = (x_\alpha)_{\alpha\in\Lambda}$, and therefore $\varphi_\alpha(\Phi(x)) = x_\alpha = \rho_\alpha(x)$. This means that the following diagram commutes, that is, $\varphi_\alpha \circ \Phi = \rho_\alpha$.



Since ρ_α is surjective, then φ_α is surjective too. This means that this projective system is perfect, as claimed. □

Corollary 6. *Every complete locally m-pseudo-convex algebra is ps-perfect with respect to its Arens-Michael decomposition. In particular, any complete locally m-convex algebra is ps-perfect with respect to its Arens-Michael decomposition.*

Proof. The completeness hypothesis implies that Φ is an isomorphism, that is, $A \cong \varprojlim A_\alpha$. □

4 ps-Perfect Algebras

In this section we show that the *ps*-perfectness property is preserved under taking finite products. One can ask if there are some other stability properties related to this notion. For instance, is a subalgebra, or a quotient algebra of a *ps*-perefect algebra (with respect to some projective system) again a *ps*-perfect algebra (with respect to some projective system)? Another question is: if $\{A_i\}_{i\in I}$ is a family of *ps*-perfect algebras (with respect to some projective systems), is the direct product, or the direct sum, the projective limit, the (strict) inductive limit of the family again *ps*-perfect (with respect to some projective system)?

If the involved algebras are complete locally m-convex algebras, then, due to Corollary 6, several of these questions are answered in the positive (see, for instance, [11, p. 81 and p. 82, Lemma 1.1]. Moreover, the tensor product algebra of two locally m-convex algebras is a topological algebra of the same type, when it is endowed with a (locally m-convex) compatible topology (see [11, p. 378, Proposition 3.1 and p. 375, Definition 3.1]. Thus, if we take the completion of the previous tensor product algebra, we get a complete locally m-convex algebra (see [11, p. 443, (4.11)]. So by Corollary 6, this (complete) tensor product algebra is *ps*-perfect with respect to its Arens-Michael decomposition.

Concerning quotient algebras, by [6, p. 41, Theorem 3.14, Definition 3.12 and the comments that follow], if A is a complete locally m-convex algebra and I is a closed ideal of A, then the quotient topological algebra A/I is a complete locally m-convex algebra. Again, by Corollary 6, it is a *ps*-perfect algebra with respect to its Arens-Michael decomposition.

Proposition 7. *If A and B are ps-perfect topological algebras with respect to the projective systems $(\{A_\alpha\}_{\alpha\in I}, \{f_{\alpha\beta}\}_{\alpha\leq\beta})$, and $(\{B_\gamma\}_{\gamma\in J}, \{g_{\gamma\delta}\}_{\gamma\leq\delta})$, respectively, then $A \times B$ is a ps-perfect topological algebra too (endowed with the product topology) with respect to the "product projective system".*

Proof. Let us take two perfect projective systems of topological algebras $(\{A_\alpha\}_{\alpha\in I}, \{f_{\alpha\beta}\}_{\alpha\leq\beta})$ and $(\{B_\gamma\}_{\gamma\in J}, \{g_{\gamma\delta}\}_{\gamma\leq\delta})$ such that $A = \varprojlim A_\alpha$ and $B = \varprojlim B_\gamma$ and let us denote by f_α and g_γ the corresponding homomorphisms from the limit to the factors, reminding that they are surjective by hypothesis.

Then we can consider the product projective system in the following way:
First, define a directed partial ordering $\leq$ in $I \times J$ in the natural way:

$$(\alpha,\gamma) \leq (\beta,\delta) \iff \alpha \leq \beta \text{ and } \gamma \leq \delta.$$

Then consider the family of topological algebras $\{A_\alpha \times B_\gamma\}_{(\alpha,\gamma)\in I\times J}$ and the family of homomorphisms

$$\begin{array}{rcl} h_{(\alpha,\gamma)(\beta,\delta)} & : & A_\beta \times B_\delta \to A_\alpha \times B_\gamma \\ (x_\beta, y_\delta) & \longmapsto & (f_{\alpha\beta}(x_\beta), g_{\gamma\delta}(y_\delta)) \end{array}$$

if $(\alpha,\gamma) \leq (\beta,\delta)$.

A straightforward verification shows that

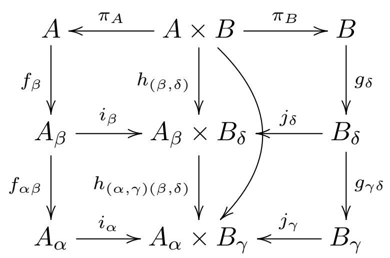
$$(\{A_\alpha \times B_\gamma\}_{(\alpha,\gamma)\in I\times J}, \{h_{(\alpha,\gamma)(\beta,\delta)}\}_{(\alpha,\gamma)\leq(\beta,\delta)})$$

is a projective system of topological algebras. We claim that its limit is isomorphic to $A \times B$.

For, let us denote by π_A and π_B the canonical projections from $A \times B$ to A and B, respectively, by i_α and j_γ the canonical inclusions from A_α and B_γ to $A_\alpha \times B_\gamma$, respectively, and by $h_{(\alpha,\gamma)}$ the homomorphisms

$$\begin{array}{rcl} h_{(\alpha,\gamma)} & : & A \times B \to A_\alpha \times B_\gamma \\ (x,y) & \longmapsto & h_{(\alpha,\gamma)}(x,y) = ((i_\alpha \circ f_\alpha \circ \pi_A)(x), (j_\gamma \circ g_\gamma \circ \pi_B)(y)) \end{array}$$

It follows that if $(\alpha,\gamma) \leq (\beta,\delta)$, then $h_{(\alpha,\gamma)(\beta,\delta)} \circ h_{(\beta,\delta)} = h_{(\alpha,\gamma)}$ (see the diagram, where the curved arrow is $h_{(\alpha,\gamma)}$).



Now, if $(U, \{k_{(\alpha,\gamma)}\}_{(\alpha,\gamma)\in I\times J}\}$ is another pair, where U is a topological algebra and the homomorphisms

$$k_{(\alpha,\gamma)} : U \to A_\alpha \times B_\gamma$$

satisfy the property that

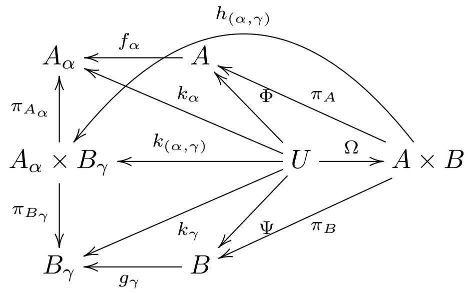
$$h_{(\alpha,\gamma)(\beta,\delta)} \circ k_{(\beta,\delta)} = k_{(\alpha,\gamma)} \text{ for each } (\alpha,\gamma) \leq (\beta,\delta),$$

consider the homomorphisms $k_\alpha : U \to A_\alpha$ and $k_\gamma = U \to B_\gamma$ defined by $k_\alpha = \pi_\alpha \circ k_{(\alpha,\gamma)}$ and $k_\gamma = \pi_\gamma \circ k_{(\alpha,\gamma)}$, respectively, where π_α and π_γ denote the canonical projections from $A_\alpha \times B_\gamma$ to A_α and B_γ, respectively.

Then we have two pairs $(U, \{k_\alpha\}_{\alpha\in I})$ and $(U, \{k_\gamma\}_{\gamma\in J})$ such that $f_{\alpha\beta} \circ k_\beta = k_\alpha$ if $\alpha \leq \beta$ and $g_{\gamma\delta} \circ k_\delta = k_\gamma$ if $\gamma \leq \delta$. Due to the universal property of the limits A and B, there exist unique homomorphisms $\Phi : U \to A$ and $\Psi : U \to B$ such that $f_\alpha \circ \Phi = k_\alpha$ and $g_\gamma \circ \Psi = k_\gamma$ for each $\alpha \in I$ and $\gamma \in J$ (see the diagram). It is clear from this that the homomorphism

$$\begin{array}{rcl} \Omega & : & U \to A \times B \\ & : & z \longmapsto (\Phi(z), \Psi(z)) \end{array}$$

satisfies that $h_{(\alpha,\gamma)} \circ \Omega = k_{(\alpha,\gamma)}$ for each $(\alpha,\gamma) \in I \times J$. The uniqueness of Ω follows from the uniqueness of Φ and Ψ. The claim is proved.



Finally, let us note that, since f_α and g_γ are surjective, then $h_{(\alpha,\gamma)}$ is also surjective (for each $(\alpha,\gamma) \in I \times J$). This proves that the projective system $(\{A_\alpha \times B_\gamma\}_{(\alpha,\gamma)\in I\times J}, \{h_{(\alpha,\gamma)(\beta,\delta)}\}_{(\alpha,\gamma)\leq(\beta,\delta)})$ is perfect and so, the topological algebra $A \times B$ is *ps*-perfect with respect to the product projective system. □

Corollary 8. *A finite direct product (sum) of ps-perfect topological algebras (with respect to some projective systems) is a ps-perfect topological algebra (with respect to some projective system).*

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